INTEGRABILITY OF ALMOST COMPLEX
STRUCTURES ON BANACH MANIFOLDS

DANIEL BELTIŢĂ

ABSTRACT. We prove that the classical integrability condition for almost complex structures on finite-dimensional smooth manifolds also works in infinite dimensions in the case of almost complex structures that are real analytic on real analytic Banach manifolds. As an application, we extend some known results concerning existence of invariant complex structures on homogeneous spaces of Banach-Lie groups.
2000 Mathematics Subject Classification: Primary 32Q60; Secondary 53C15;58B12
Key words and phrases: almost complex structure; Banach-Lie group; homogeneous space

1. INTRODUCTION

In this paper we prove that the classical integrability condition for almost complex structures on finite-dimensional smooth manifolds (see [EF51], [Fr55], [NN57]) also works in infinite dimensions, ensuring the existence of complex coordinates in the case of almost complex structures that are real analytic on real analytic Banach manifolds. And this might turn out to be interesting in connection with both the recent interest in fundamental problems in infinite-dimensional complex analysis (see [Le98], [Le99] and [Le00]) and the role of infinite-dimensional complex homogeneous spaces in representation theory of infinite-dimensional Lie groups (see for instance [NRW01] and [Ne04]).

Our motivation in writing this paper came from the fact that, in the existing literature, the subject of complex coordinates in almost complex Banach manifolds was approached in two very different manners: either one constructed the complex coordinates in a way that seemed to be very specific to the manifolds under consideration (see for instance [Ne04], [BR04], [Be03], [MS98], [MS97], [Wi94] and [Wi90]) or one exhibited examples of almost complex structures that do not come from any complex coordinates at all (see [Pa00]). Our point is to show that there exists a large class of almost complex structures, namely the torsion-free real analytic almost complex structures on real analytic Banach manifolds, that always come from complex coordinates on the corresponding manifold (Theorem 7 below). That this class of almost complex structures is really wide can be seen from the fact that all invariant almost complex structures on homogeneous spaces of Banach-Lie groups are real analytic (see Proposition 11 below).

As a matter of fact, our main result on invariant complex structures (Theorem 15) works for homogeneous spaces of not necessarily connected Banach-Lie groups. This point is particularly important for the applications to the adjoint or coadjoint orbits that show up in connection with the operator algebras (see [BR04], [MS98], [MS97] and Corollary 16 below), since the unitary groups of C^*-algebras are

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

in general non-connected (although they are connected in the special case of W^*-algebras), while nothing seems to be known on connectedness of the unitary groups associated with non-separable operator ideals (compare Remark 4.9 in [BR04]). Thus, our method improves the one suggested in [Ne04], where one needs the connectedness of the group that transitively acts on the homogeneous space. (The method suggested in [Ne04] is based on extending to infinite dimensions certain reasonings from [Ki76] involving local groups and Baker-Campbell-Hausdorff series.)

The basic idea in the proofs of our main results is to make sure that the main steps in the finite-dimensional original reasonings (see [Fr55]) carry over to infinite dimensions. In order to do so, a few technical devices are needed from infinite-dimensional global analysis (see [Up85], [Nu92] and [La01]). The paper is organized as follows: In Section 2 we prove the main result on almost complex structures on general manifolds (Theorem 7). In Section 3 we consider the case of homogeneous spaces of Banach-Lie groups, and the main result of this section describes a way to parameterize the invariant complex structures in Lie algebraic terms (Theorem 15).

We now introduce some notation that will be used throughout the paper. We denote by i the square root of -1, by e the basis of the natural logarithms, by TM the tangent bundle of a manifold M and by $B_X(a, r)$ the open ball of center $a \in X$ and radius $r > 0$ in the real Banach space X. Moreover $\mathcal{B}(X)$ stands for the bounded real-linear operators on the real Banach space X, while $\mathcal{B}^{\mathbb{C}}(Z)$ denotes the bounded complex-linear operators on a complex Banach space Z. As regards function spaces, $\mathcal{C}^{\infty}(D, Y)$ is the set of $\mathcal{C}^{\infty}$ functions on the open subset D of some real Banach space with values in another real Banach space Y, $\mathcal{C}^{\infty}(D) := \mathcal{C}^{\infty}(D, \mathbb{R})$ $\mathcal{C}^{\omega}(D, Y)$ is the space of real analytic functions on the open subset D of some real Banach space with values in the real Banach space Y, $\mathcal{O}(D, W)$ is the space of holomorphic functions on the open subset D of some complex Banach space with values in another complex Banach space W, and $\mathcal{O}(D) := \mathcal{O}(D, \mathbb{C})$.

We denote by $X_{\mathbb{C}} = X \dotplus \mathrm{i}X$ the complexified Banach space of the real Banach space X. We always think of $X_{\mathbb{C}}$ equipped with a norm such that $\|x - \mathrm{i}y\| = \|x + \mathrm{i}y\|$ for all $x, y \in X$. Moreover, unless otherwise stated, we denote by $z \mapsto \overline{z}$ the conjugation of Z whose set of fixed points equals X, that is, $\overline{x + \mathrm{i}y} = x - \mathrm{i}y$ whenever $x, y \in X$. We also note that every $T \in \mathcal{B}(X)$ can be uniquely extended to an element of $\mathcal{B}^{\mathbb{C}}(X_{\mathbb{C}})$ which will always be denoted by T as well.

For an open subset D of a real Banach space X, we consider the usual Lie bracket $[\cdot, \cdot]$ defined on $\mathcal{C}^{\infty}(D, X)$ by

$$[a, b]_x = a'_x(b_x) - b'_x(a_x)$$

whenever $x \in D$ and $a, b \in \mathcal{C}^{\infty}(D, X)$, where we denote the value of a at x by $a_x := a(x) \in X$, the differential of a at x by $a'_x : X \to X$, the second differential of a at x by $a''_x : X \times X \to X$, and so on. Also, for a smooth mapping $J : D \to \mathcal{B}(X)$ and any $a \in \mathcal{C}^{\infty}(D, X)$ we define $Ja \in \mathcal{C}^{\infty}(D, X)$ by $(Ja)_x = J_x a_x$ for all $x \in D$.

2. GENERAL RESULTS

Definition 1. Let M be a smooth real Banach manifold. An *almost complex structure* on M is a mapping $J : TM \to TM$ such that for all $p \in M$ we have $J(T_pM) \subseteq T_pM$, $J_p := J|_{T_pM} : T_pM \to T_pM$ is a bounded (real-)linear operator and $(J_p)^2 = -\mathrm{id}_{T_pM}$.

Lemma 2. *Let X be a real Banach space and D an open subset of X. Suppose that*

$$J: D \to \mathcal{B}(X), \quad x \mapsto J_x$$

is a smooth mapping satisfying $(J_x)^2 = -\mathrm{id}_X$ for all $x \in D$, and consider

$$\Omega: \mathcal{C}^\infty(D, X) \times \mathcal{C}^\infty(D, X) \to \mathcal{C}^\infty(D, X)$$

defined by

$$\Omega(a, b) = J[Ja, b] + J[a, Jb] + [a, b] - [Ja, Jb]$$

for all $a, b \in \mathcal{C}^\infty(D, X)$. Then we have

$$\Omega(a, b)_x = -J'_x(J_x a_x, b_x) - J'_x(a_x, J_x b_x) + J'_x(J_x b_x, a_x) + J'_x(b_x, J_x a_x)$$

for all $a, b \in \mathcal{C}^\infty(D, X)$ and $x \in D$.

Proof. The proof is a straightforward computation based on the formulas

$$(Jc)'_x u = J'_x(c_x, u) + J_x c'_x u \quad \text{whenever } x \in D, u \in X \text{ and } c \in \mathcal{C}^\infty(D, X), \quad (1)$$

and

$$J'_x(J_x u, v) + J_x J'_x(u, v) = 0 \quad \text{whenever } x \in D \text{ and } u, v \in X. \quad (2)$$

We omit the details. $\square$

Definition 3. Let M be a smooth real Banach manifold, denote by $TM \oplus TM$ the Whitney sum of the vector bundle TM with itself and assume that $J: TM \to TM$ is a smooth almost complex structure. Then the *torsion* of J is the smooth mapping

$$\Omega: TM \oplus TM \to TM$$

having the property that, if a and b are locally defined smooth vector fields on M, then we have

$$\Omega(a, b) = J[Ja, b] + J[a, Jb] + [a, b] - [Ja, Jb]$$

on any open set where both a and b are defined. Note that the existence of the mapping Ω is a consequence of Lemma 2.

Lemma 4. *Let X be a real Banach space, $r > 0$ and $J: B_X(0, r) \to \mathcal{B}(X)$ a real analytic mapping satisfying $(J_x)^2 = -\mathrm{id}_X$ whenever $x \in B_X(0, r)$. Consider the complexified Banach space $Z = X_\mathbb{C}$ and extend $J|_{B_X(0, r/2\mathrm{e})}$ to a holomorphic mapping $J: B_Z(0, r/2\mathrm{e}) \to \mathcal{B}^\mathbb{C}(Z)$. Next denote*

$$\mathcal{Z}_\pm := \{a \in \mathcal{O}(B_Z(0, r/2\mathrm{e}), Z) \mid Ja = \mp \mathrm{i}a\}$$

and define

$$\Omega: \mathcal{C}^\infty(D, Y) \times \mathcal{C}^\infty(D, Y) \to \mathcal{C}^\infty(D, Y)$$

as in Lemma 2, where Y stands for either of the spaces X and Z, and D is any open subset of the corresponding Banach space (Z is thought of as a real Banach space).

Now consider the following assertions:
1° For all $a, b \in \mathcal{C}^\omega(B_X(0, r/2\mathrm{e}), X)$ we have $\Omega(a, b) = 0$.

$2°$ *For all $a, b \in \mathcal{Z}_-$ we have $[a, b] \in \mathcal{Z}_-$.*
$3°$ *For all $a, b \in \mathcal{Z}_+$ we have $[a, b] \in \mathcal{Z}_+$.*
$4°$ *For all $a, b \in \mathcal{C}^\omega(B_X(0, r), X)$ we have $\Omega(a, b) = 0$.*
Then we have $1° \Rightarrow 2° \Leftrightarrow 3° \Rightarrow 2°$.

Proof. We first note that, if $n \in \mathbb{Z}_+$, $f_n \colon Z \times \cdots \times Z \to Z$ is complex n-linear and $f_n(X \times \cdots \times X) \subseteq X$, then

$$\|f_n(x_1 + \mathrm{i}y_1, \ldots, x_n + \mathrm{i}y_n)\| \leq \|f_n|_{X \times \cdots \times X}\| \cdot (\|x_1\| + \|y_1\|) \cdots (\|x_n\| + \|y_n\|)$$
$$\leq 2^n \|f_n|_{X \times \cdots \times X}\| \cdot (\|x_1 + \mathrm{i}y_1\|) \cdots (\|x_n + \mathrm{i}y_n\|),$$

hence $\|f_n\| \leq 2^n \|f_n|_{X \times \cdots \times X}\|$. It follows by this remark along with Corollary 1.5 in [Up85] that for each $a \in \mathcal{C}^\omega(B_X(0, r), X)$ there is a mapping in $\mathcal{O}(B_Z(0, r/2\mathrm{e}), Z)$ that agrees with a on $B_X(0, r/2\mathrm{e})$; that holomorphic mapping is uniquely determined according to Theorem 1.11 in [Up85], and therefore it will be denoted by a as well. Similarly we define $J \colon B_Z(0, r/2\mathrm{e}) \to \mathcal{B}^{\mathbb{C}}(Z)$. Throughout the proof, we also define

$$\Omega \colon \mathcal{O}(B_Z(0, r/2\mathrm{e}), Z) \times \mathcal{O}(B_Z(0, r/2\mathrm{e}), Z) \to \mathcal{O}(B_Z(0, r/2\mathrm{e}), Z)$$

by the same formula as in Lemma 2. Now we come back to the proof of the asserted relationship between assertions $1°$–$4°$.

$1° \Rightarrow 2°$ Let $a, b \in \mathcal{Z}_-$, so that $Ja = -\mathrm{i}a$ and $Jb = -\mathrm{i}b$. By $1°$ we have $\Omega(a, b)|_{B_X(0, r/2\mathrm{e})} = 0$, hence $\Omega(a, b) = 0$ on $B_Z(0, r/2\mathrm{e})$ by Theorem 1.11 in [Up85]. Then

$$0 = J[-\mathrm{i}a, b] + J[a, -\mathrm{i}b] + [a, b] - [-\mathrm{i}a, -\mathrm{i}b] = -2\mathrm{i}J[a, b] + 2[a, b],$$

whence $J[a, b] = -\mathrm{i}[a, b]$ as desired.

$2° \Rightarrow 3°$ If $a, b \in \mathcal{Z}_-$, then clearly $\overline{a(\cdot)}, \overline{b(\cdot)} \in \mathcal{Z}_+$, hence $[\overline{a(\cdot)}, \overline{b(\cdot)}] \in \mathcal{Z}_+$. Now $\overline{[a, b](\cdot)} = [\overline{a(\cdot)}, \overline{b(\cdot)}] \in \mathcal{Z}_+$, whence $[a, b] \in \mathcal{Z}_-$.

$3° \Rightarrow 2°$ Similar to $2° \Rightarrow 3°$.

$3° \Rightarrow 4°$ Let $a, b \in \mathcal{C}^\omega(B_X(0, r), X)$ arbitrary and construct the corresponding $a, b \in \mathcal{O}(B_Z(0, r/2\mathrm{e}), Z)$ as explained at the beginning of the proof. Then we have

$$a_\pm := \frac{1}{2}(a \mp \mathrm{i}Ja) \in \mathcal{Z}_\pm, \quad b_\pm := \frac{1}{2}(b \mp \mathrm{i}Jb) \in \mathcal{Z}_\pm,$$

and

$$a = a_+ + a_- \text{ and } b = b_+ + b_- \text{ on } B_Z(0, r/2\mathrm{e}).$$

In particular,

$$\Omega(a, b) = \Omega(a_+, b_+) + \Omega(a_+, b_-) + \Omega(a_-, b_+) + \Omega(a_-, b_-) \text{ on } B_Z(0, r/2\mathrm{e}).$$

Now, since $Ja_\pm = \mp \mathrm{i}a_\pm$ and $Jb_\pm = \mp \mathrm{i}b_\pm$, we easily get by the definition of $\Omega(\cdot, \cdot)$ that $\Omega(a, b) = 0$ on $B_Z(0, r/2\mathrm{e})$. Then Theorem 1.11 in [Up85] shows that $\Omega(a, b) = 0$ on $B_X(0, r)$, and the proof is finished. $\square$

Lemma 5. *In the setting of* Lemma 4, *denote*

$$Z_0^- := \operatorname{Ker}(J_0 + \mathrm{i} \cdot \operatorname{id}_Z) \quad (\subseteq Z).$$

Then there exists $r_1 \in (0, r/2\mathrm{e})$ such that for every $z \in B_Z(0, r_1)$ the sequence

$$0 \to Z_0^- \xrightarrow{J_z - \mathrm{i} \cdot \operatorname{id}_Z} Z \xrightarrow{J_z + \mathrm{i} \cdot \operatorname{id}_Z} Z$$

is exact.

Proof. To begin with, denote $Z_z^- := \operatorname{Ker}(J_z + \mathrm{i} \cdot \operatorname{id}_Z) \subseteq Z$ for all $z \in B_Z(0, r/2\mathrm{e})$. Since $(J_z)^2 = -\operatorname{id}_Z$, it then follows that for all $z \in B_Z(0, r/2\mathrm{e})$ we have

$$Z_z^- = \operatorname{Ran}(J_z - \mathrm{i} \cdot \operatorname{id}_Z) \text{ and } Z = Z_z^- \dotplus \operatorname{Ker}(J_z - \mathrm{i} \cdot \operatorname{id}_Z).$$

In particular, it follows that $(J_0 - \mathrm{i} \cdot \operatorname{id}_Z)|_{Z_0^-} : Z_0^- \to Z_0^-$ is an isomorphism of the complex Banach space Z_0^- onto itself.

On the other hand, for arbitrary $z \in B_Z(0, r)$, note that we have an isomorphism of real Banach spaces

$$\Psi_z : Z_z^- \to X, \quad z \mapsto \frac{1}{2}(z + \overline{z}).$$

In fact, since $\overline{J_z v} = J_z \overline{v}$ for all $v \in Z$, it is easy to check that Ψ_z has an inverse defined by $x \mapsto x + \mathrm{i} J_z x$.

Now we have a continuous mapping

$$\tau : B_Z(0, r) \to \mathcal{B}(Z_0^-, X), \quad z \mapsto \Psi_z \circ (J_z - \mathrm{i} \cdot \operatorname{id}_Z)|_{Z_0^-},$$

and this mapping has the property that $\tau(0) : Z_0^- \to X$ is an isomorphism, since both $(J_0 - \mathrm{i} \cdot \operatorname{id}_Z)|_{Z_0^-} : Z_0^- \to Z_0^-$ and $\Psi_0 : Z_0^- \to X$ are isomorphisms. Since the isomorphisms $Z_0^- \to X$ constitute an open subset of $\mathcal{B}(Z_0^-, X)$, it then follows that there exists $r_1 \in (0, r/2\mathrm{e})$ such that $\tau(z) : Z_0^- \to X$ is an isomorphism of real Banach spaces whenever $z \in B_Z(0, r_1)$. Since each $\Psi_z : Z_z^- \to X$ is an isomorphism, we deduce that

$$(J_z - \mathrm{i} \cdot \operatorname{id}_Z)|_{Z_0^-} : Z_0^- \to Z_z^- = \operatorname{Ker}(J_z + \mathrm{i} \cdot \operatorname{id}_Z)$$

is an isomorphism for all $z \in B_Z(0, r_1)$, and the proof ends. $\square$

Theorem 6. *Let M be a complex Banach manifold and E a complex subbundle of TM with the following property: if V is an open subset of M and $\xi, \eta : V \to TM$ are holomorphic vector fields on V such that $\xi(p), \eta(p) \in E_p$ for all $p \in V$, then $[\xi, \eta](p) \in E_p$ for all $p \in V$. Then for every $m \in M$ there exist an open neighborhood U of m, a complex Banach space F and a holomorphic submersion $\zeta : U \to F$ such that*

$$E_p = \operatorname{Ker}(T_p\zeta) = T_p(\zeta^{-1}(\zeta(p))) \text{ for all } p \in U.$$

Proof. All the needed reasonings that are carried out in Chapter VI in [La01] in order to prove the Frobenius theorem for $\mathcal{C}^k$ mappings on real Banach spaces work for holomorphic mappings on complex Banach spaces as well. $\square$

Now we are ready to prove the following infinite-dimensional version of the result of §11 in [Fr55]. The proof is inspired by the last remark in §16 in [Fr55].

Theorem 7. *Let C be a real analytic Banach manifold and*

$$J : TC \to TC$$

a real analytic almost complex structure on C. Denote by

$$\Omega : TC \oplus TC \to TC$$

the torsion of J. Then $\Omega = 0$ if and only if there exists on C a structure of complex Banach manifold whose underlying real analytic structure is just the original structure of C and such that $J_p v = \mathrm{i}v$ whenever $p \in C$ and $v \in T_p C$.

Proof. The proof has several stages.

1° It is clear that $\Omega = 0$ is a necessary condition for the existence of a complex structure as indicated in the statement.

2° Conversely, assume that $\Omega = 0$. Since the statement has a local character, it suffices to prove the following fact. Let X be a real Banach space, $r > 0$ and

$$J : B_X(0,r) \to \mathcal{B}(X), \quad x \mapsto J_x,$$

a real analytic mapping such that $(J_x)^2 = -\mathrm{id}_X$ for all $x \in B_X(0,r)$ and

$$J[a, Jb] + J[Ja, b] + [a, b] - [Ja, Jb] = 0 \quad \text{for all } a, b \in \mathcal{C}^\omega(B_X(0,r), X).$$

Then there exist $r_0 \in (0, r)$, a complex manifold N and a real analytic diffeomorphism $\Theta : B_X(0, r_0) \to N$ such that

$$T_x\Theta \circ J_x = \mathrm{i}T_x\Theta \quad \text{for all } x \in B_X(0, r_0).$$

3° To prove the assertion from stage 2°, let Z, $J : B_Z(0, r/2\mathrm{e}) \to \mathcal{B}^{\mathbb{C}}(Z)$ and $\mathcal{Z}_\pm$ as in Lemma 4. Moreover denote $M = B_Z(0, r/2\mathrm{e})$ and

$$E_z = \{z\} \times \mathrm{Ker}\,(J_z + \mathrm{i} \cdot \mathrm{id}_Z) \subseteq M \times Z \quad \text{for all } z \in M.$$

It then follows by Lemma 5 that $E = \bigcup_{z \in M} E_z$ is a complex subbundle of TM (see [La01]).

On the other hand, Lemma 4 ($1° \Rightarrow 3°$) shows that, if V is an open subset of M and $\xi, \eta : V \to TM$ are holomorphic vector fields with $\xi(p), \eta(p) \in E_p$ for all $p \in V$. then $[\xi, \eta](p) \in E_p$ whenever $p \in V$. Now it follows by Theorem 6 that there exist $r_0 \in (0, r/2\mathrm{e})$, a complex Banach space F and a holomorphic submersion $\zeta : B_Z(0, r_0) \to F$ such that

$$\mathrm{Ker}\,(T_z\zeta) = E_z = \{z\} \times \mathrm{Ker}\,(J_z + \mathrm{i} \cdot \mathrm{id}_Z) \quad \text{for all } z \in B_Z(0, r_0).$$

In particular, since we have $\mathrm{Ran}\,(J_z - \mathrm{i} \cdot \mathrm{id}_Z) = \mathrm{Ker}\,(J_z + \mathrm{i} \cdot \mathrm{id}_Z)$, it follows that $T_z\zeta \circ (J_z - \mathrm{i} \cdot \mathrm{id}_Z) = 0$, whence

$$T_z\zeta \circ J_z = \mathrm{i}T_z\zeta \quad \text{for all } z \in B_Z(0, r_0).$$

4° We now show that

$$\mathrm{Ker}\,(J_0 + \mathrm{i} \cdot \mathrm{id}_Z) \dotplus X = Z.$$

In fact, since $J_0 X = X$, it is clear that $\mathrm{Ker}\,(J_0 + \mathrm{i} \cdot \mathrm{id}_Z) \cap X = \{0\}$. Now let $z \in Z$ arbitrary. Then $z_\pm := (J_0 \pm \mathrm{i} \cdot \mathrm{id}_Z)z/2\mathrm{i} \in \mathrm{Ker}\,(J_0 \mp \mathrm{i} \cdot \mathrm{id}_Z)$ and $z = z_+ - z_-$. Thus $z = (-z_- - \overline{z_+}) + (z_+ + \overline{z_+})$ with $-z_- - \overline{z_+} \in \mathrm{Ker}\,(J_0 + \mathrm{i} \cdot \mathrm{id}_Z)$ and $z_+ + \overline{z_+} \in X$.

Since $\mathrm{Ker}\,(T_0\zeta) = \{0\} \times \mathrm{Ker}\,(J_0 + \mathrm{i} \cdot \mathrm{id}_Z)$ and $\{0\} \times X = T_0(B_X(0, r_0))$, it then follows that $T_0(\zeta|_{B_X(0,r_0)}) = (T_0\zeta)|_X \colon X \to \mathrm{Ran}\,(T_0\zeta) = F$ is an isomorphism of real Banach spaces. (Recall that ζ is a submersion.) The inverse mapping theorem then shows that there exist $r_1 \in (0, r_0)$ and an open neighborhood N of $\zeta(0) \in F$ such that $\Theta := \zeta|_{B_X(0,r_1)} \colon B_X(0, r_1) \to N$ is a real analytic diffeomorphism. Now the last assertion in stage $3°$ shows that the diffeomorphism Θ has all the properties claimed in stage $2°$ of the proof. $\square$

3. The case of homogeneous spaces

Notation 8. Throughout this section we denote by G a real Banach-Lie group G and by H a Banach-Lie subgroup of G (see for instance [Up85]). Then G/H has a structure of real analytic manifold such that the natural action

$$\alpha \colon G \times G/H \to G/H, \quad (g, p) \mapsto \alpha(g, p) = \alpha_g(p) = g.p$$

is real analytic. Furthermore, we consider the natural projection

$$\pi \colon G \to G/H,$$

which is a principal bundle. We denote the base point of G/H by

$$p_0 := H \in G/H.$$

The Lie algebras of G and H will be denoted by $\mathbf{L}(G) = \mathfrak{g}$ and $\mathbf{L}(H) = \mathfrak{h}$ respectively. Then we have a natural identification

$$T_{p_0}(G/H) \simeq \mathfrak{g}/\mathfrak{h}.$$

Also note that the restriction to H of the adjoint representation $\mathrm{Ad}_G \colon G \to \mathrm{Aut}(\mathfrak{g})$ induces a continuous homomorphism of Banach-Lie groups

$$\widetilde{\mathrm{Ad}}_G \colon H \to \mathrm{GL}(\mathfrak{g}/\mathfrak{h}) = \mathrm{GL}(T_{p_0}(G/H)).$$

Definition 9. Let $J \colon T(G/H) \to T(G/H)$ be an almost complex structure on the homogeneous space G/H. We say that J is *invariant* if for all $g \in G$ and $p \in G/H$ the diagram

$$
\begin{array}{ccc}
T_p(G/H) & \xrightarrow{T_p(\alpha_g)} & T_{g.p}(G/H) \\
{\scriptstyle J_p}\big\downarrow & & \big\downarrow{\scriptstyle J_{g.p}} \\
T_p(G/H) & \xrightarrow{T_p(\alpha_g)} & T_{g.p}(G/H)
\end{array}
$$

is commutative.

Proposition 10. *Consider the sets*

$$\mathcal{I}ac(G,H) = \{J : T(G/H) \to T(G/H) \mid J \text{ invariant almost complex structure}\}$$

and

$$\mathcal{I}_0(G,H) = \{I \in \mathcal{B}(\mathfrak{g}/\mathfrak{h}) \mid I^2 = -\mathrm{id}_{\mathfrak{g}/\mathfrak{h}}; \quad (\forall h \in H) \ T_{p_0}(\alpha_h) \circ I = I \circ T_{p_0}(\alpha_h)\}.$$

Then the mappings

$$\check{\ }: \mathcal{I}ac(G,H) \to \mathcal{I}_0(G,H), \quad J \mapsto \check{J} := J_{p_0},$$

and

$$\widehat{\ }: \mathcal{I}_0(G,H) \to \mathcal{I}ac(G,H), \quad \widehat{I}_{g.p_0} := T_{p_0}(\alpha_g) \circ I \circ T_{p_0}(\alpha_g)^{-1} \text{ for all } g \in G,$$

are bijections inverse to one another.

Proof. For each $I \in \mathcal{I}_0(G,H)$ the mapping $\widehat{I}_{g.p_0} : T_{g.p_0}(G/H) \to T_{g.p_0}(G/H)$ is well defined for all $g \in G$. Thus the mapping $\widehat{\ }$ is well defined. Now the assertions are obvious. □

Proposition 11. *Every invariant almost complex structure J on G/H is real analytic.*

Proof. This follows by an easy application of Corollary 8.4 in [Up85]. □

The following result is a more precise version of Satz 2 of §18 in [Fr55] in infinite dimensions.

Proposition 12. *Let $\mathcal{V}$ be a closed linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \dotplus \mathcal{V}$ and $\mathcal{I}_\mathcal{V}(G,H)$ the set of all bounded linear operators $I : \mathfrak{g} \to \mathfrak{g}$ such that $\mathfrak{h} \subseteq \mathrm{Ker}\, I$, $I(\mathcal{V}) \subseteq \mathcal{V}$, $(I|_\mathcal{V})^2 = -\mathrm{id}_\mathcal{V}$ and $T_\mathbf{1}\pi \circ I \circ \mathrm{Ad}_G(h)|_\mathcal{V} = T_\mathbf{1}\pi \circ \mathrm{Ad}_G(h) \circ I|_\mathcal{V}$ for all $h \in H$. Now for every $I \in \mathcal{I}_\mathcal{V}(G,H)$ define $c_\mathcal{V}(I) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h}$ by $c_\mathcal{V}(I)(x + \mathfrak{h}) = Ix + \mathfrak{h}$ for all $x \in \mathfrak{g}$.*

Then we obtain a bijection

$$c_\mathcal{V} : \mathcal{I}_\mathcal{V}(G,H) \to \mathcal{I}_0(G,H).$$

Proof. The proof follows the lines of the proof of Satz 2 of §18 in [Fr55], so that we omit the details. □

We now prove an extension of Satz 2 of §19 in [Fr55].

Theorem 13. *Let J be an invariant almost complex structure on G/H and $\mathcal{V}$ a closed linear subspace of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \dotplus \mathcal{V}$. Now let $I \in \mathcal{I}_\mathcal{V}(G,H)$ $(\subseteq \mathcal{B}(\mathfrak{g}))$ with $c_\mathcal{V}(I) = J_{p_0}$. Then the following assertions are equivalent:*
  (i) *There exists on G/H a structure of complex Banach manifold whose underlying real analytic structure is just the original structure of G/H and such that $J_p v = \mathrm{i}v$ whenever $p \in G/H$ and $v \in T_p(G/H)$.*
  (ii) *For all $x, y \in \mathfrak{g}$ we have $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] \in \mathfrak{h}$.*

Proof. We begin with a few general remarks. First note that the existence of I in the statement follows by Propositions 10 and 12. Next, for $x \in \mathfrak{g}$ arbitrary, we

introduce the notation $\widetilde{x}$ for the corresponding G-invariant (real analytic) vector field on G/H. Note that for each $x \in \mathfrak{g}$ we have

$$T_{\mathbf{1}}\pi(x) = \widetilde{x}_{p_0} \text{ and } \widetilde{Ix} = J\widetilde{x}. \tag{3}$$

Moreover, since $I \in \mathcal{I}_{\mathcal{V}}(G, H)$ (see Proposition 12), we have $T_{\mathbf{1}}\pi \circ I \circ \mathrm{Ad}_G(h)|_{\mathcal{V}} = T_{\mathbf{1}}\pi \circ \mathrm{Ad}_G(h) \circ I|_{\mathcal{V}}$ for all $h \in H$. By differentiating this equality with respect to $h \in H$, we see that

$$T_{\mathbf{1}}\pi(I[z, v]) = T_{\mathbf{1}}\pi[z, Iv] \text{ whenever } z \in \mathfrak{h} \text{ and } v \in \mathcal{V}.$$

Also, recall from the commutative diagram in the proof of Proposition 12 that $T_{\mathbf{1}}\pi \circ I = J_{p_0} \circ T_{\mathbf{1}}\pi$. This implies that for all $x, y \in \mathcal{V}$ we have

$$T_{\mathbf{1}}\pi(I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy])$$
$$= J_{p_0}(T_{\mathbf{1}}\pi[Ix, y]) + J_{p_0}(T_{\mathbf{1}}\pi[x, Iy]) + [\widetilde{x}, \widetilde{y}]_{p_0} - [\widetilde{Ix, Iy}]_{p_0}$$
$$= J_{p_0}[\widetilde{Ix, y}]_{p_0} + J_{p_0}[\widetilde{x, Iy}]_{p_0} + [\widetilde{x}, \widetilde{y}]_{p_0} - [\widetilde{Ix, Iy}]_{p_0}$$
$$= J_{p_0}[\widetilde{Ix}, \widetilde{y}]_{p_0} + J_{p_0}[\widetilde{x}, \widetilde{Iy}]_{p_0} + [\widetilde{x}, \widetilde{y}]_{p_0} - [J\widetilde{x}, J\widetilde{y}]_{p_0}$$
$$= (J[J\widetilde{x}, \widetilde{y}] + J[\widetilde{x}, \widetilde{Jy}] + [\widetilde{x}, \widetilde{y}] - [J\widetilde{x}, J\widetilde{y}])_{p_0},$$

where the last equality follows by (3).

We now proceed to prove that assertions (i) and (ii) are equivalent.

(i)$\Rightarrow$(ii) It follows by Theorem 7 that the torsion of J vanishes, hence the above computation shows that for all $x, y \in \mathcal{V}$ we have $T_{\mathbf{1}}\pi(I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy]) = 0$, that is, $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] \in \mathfrak{h}$.

Moreover, if $x \in \mathfrak{h}$ and $y \in \mathcal{V}$, then

$$T_{\mathbf{1}}\pi(I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy])$$
$$= T_{\mathbf{1}}\pi[Ix, Iy] + T_{\mathbf{1}}\pi[x, I^2 y] + T_{\mathbf{1}}\pi[x, y] - T_{\mathbf{1}}\pi[Ix, Iy]$$
$$= 0,$$

hence also $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] \in \mathfrak{h}$.

Finally, if $x, y \in \mathfrak{h}$ then $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] = [x, y] \in \mathfrak{h}$.

(ii)$\Rightarrow$(i) By the computation performed just before the proof of (i)$\Rightarrow$(ii), for $x, y \in \mathcal{V}$ arbitrary we get $(J[J\widetilde{x}, \widetilde{y}] + J[\widetilde{x}, \widetilde{Jy}] + [\widetilde{x}, \widetilde{y}] - [J\widetilde{x}, J\widetilde{y}])_{p_0} = 0$. Since the almost complex structure J is invariant and both $\widetilde{x}$ and $\widetilde{y}$ are G-invariant vector fields on G/H, it then follows that $J[J\widetilde{x}, \widetilde{y}] + J[\widetilde{x}, \widetilde{Jy}] + [\widetilde{x}, \widetilde{y}] - [J\widetilde{x}, J\widetilde{y}] = 0$ on G/H. But this implies that the torsion of J vanishes, since for every $p \in G/H$ we have $T_p(G/H) = \{\widetilde{x} \mid x \in \mathcal{V}\}$ in view of the fact $\pi \colon G \to G/H$ is a submersion and $\mathcal{V} \dotplus \mathrm{Ker}\,(T_{\mathbf{1}}\pi) = \mathfrak{g}$. $\square$

Definition 14. We denote by $\mathcal{I}co(G, H)$ the set of all invariant almost complex structures J on G/H such that assertion (i) in Theorem 13 holds. Each element $J \in \mathcal{I}co(G, H)$ will be called an *invariant complex structure* on the homogeneous space G/H.

Now we are able to give the following parameterization of the invariant complex structures in Lie algebraic terms. When specialized to finite dimensions, the next theorem becomes a more precise version of the Satz of §20 in [Fr55].

Theorem 15. *Denote by $\mathcal{K}_0(G,H)$ the set of all closed complex subalgebras $\mathfrak{k}$ of $\mathfrak{g}_\mathbb{C}$ such that*

$$\mathfrak{k} + \overline{\mathfrak{k}} = \mathfrak{g}_\mathbb{C}, \; \mathfrak{k} \cap \overline{\mathfrak{k}} = \mathfrak{h}_\mathbb{C} \text{ and } \mathrm{Ad}_G(h)\mathfrak{k} \subseteq \mathfrak{k} \text{ for all } h \in H.$$

Next fix a closed linear subspace $\mathcal{V}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \dotplus \mathcal{V}$ and for each $J \in \mathcal{I}co(G,H)$ define

$$b_\mathcal{V}(J) = \mathfrak{h}_\mathbb{C} + \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}}),$$

where $I := (c_\mathcal{V})^{-1}(J_{p_0}) \in \mathcal{I}_\mathcal{V}(G,H) \subseteq \mathcal{B}(\mathfrak{g})$.
    Then we obtain a bijection

$$b_\mathcal{V}:\mathcal{I}co(G,H) \to \mathcal{K}_0(G,H).$$

Proof. The proof has several stages.
    $1°$ At this stage we show that $b_\mathcal{V}$ is well defined in the sense that for $J \in \mathcal{I}co(G,H)$ arbitrary we have indeed $\mathfrak{k} := c_\mathcal{V}(J) \in \mathcal{K}_0(G,H)$. What is obvious is that $\mathfrak{k}$ is a closed complex linear subspace of $\mathfrak{g}_\mathbb{C}$.
    Since $I^2 = -\mathrm{id}_{\mathcal{V}_\mathbb{C}}$ on $\mathcal{V}_\mathbb{C}$, we get $\mathcal{V}_\mathbb{C} = \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}}) \dotplus \mathrm{Ker}\,(I + \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}})$, whence

$$\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \dotplus \mathcal{V}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \dotplus \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}}) \dotplus \mathrm{Ker}\,(I + \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}}). \tag{4}$$

Now, since $I\overline{z} = \overline{Iz}$ for all $z \in \mathfrak{g}_\mathbb{C}$ and $\mathfrak{k} = \mathfrak{h}_\mathbb{C} + \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}})$, it follows by (4) that $\mathfrak{k} + \overline{\mathfrak{k}} = \mathfrak{g}_\mathbb{C}$ and $\mathfrak{k} \cap \overline{\mathfrak{k}} = \mathfrak{h}_\mathbb{C}$.
    Furthermore let $h \in H$ arbitrary. If $v \in \mathfrak{h}_\mathbb{C}$ it is clear that $\mathrm{Ad}_g(h)v \in \mathfrak{h}_\mathbb{C}$. Now let $v \in \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_\mathcal{V})$ arbitrary. Since $I \in \mathcal{I}_\mathcal{V}(G,H)$, it follows that $(I \circ \mathrm{Ad}_G(h))v - (\mathrm{Ad}_g(h) \circ I)v \in \mathfrak{h}_\mathbb{C}$, that is, $I(\mathrm{Ad}_G(h)v) - \mathrm{i}\cdot\mathrm{Ad}_G(h)v \in \mathrm{Ker}\,I$. Consequently $0 = I(I(\mathrm{Ad}_G(h)v) - \mathrm{i}\cdot\mathrm{Ad}_G(h)v) = -\mathrm{Ad}_G(h)v - \mathrm{i}\cdot(I \circ \mathrm{Ad}_G(h))v$, whence $I(\mathrm{Ad}_G(h)v) = \mathrm{i}\cdot\mathrm{Ad}_G(h)v$, which shows that $\mathrm{Ad}_G(h)v \in \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_\mathcal{V})$. Thus $\mathrm{Ad}_G(h)\mathfrak{k} \subseteq \mathfrak{k}$ for arbitrary $h \in H$.
    Thence by differentiating with respect to $h \in H$ we get $[\mathfrak{h}, \mathfrak{k}] \subseteq \mathfrak{k}$, whence $[\mathfrak{h}_\mathbb{C}, \mathfrak{k}] \subseteq \mathfrak{k}$. Thus, in order to prove that $\mathfrak{k}$ is a subalgebra of $\mathfrak{g}_\mathbb{C}$, it remains to check that $[x,y] \in \mathfrak{k}$ for arbitrary $x,y \in \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_\mathcal{V})$. In fact, since $J \in \mathcal{I}co(G,H)$, it follows by Theorem 13((i)$\Rightarrow$(ii)) that $I[Ix,y] + I[x,Iy] + [x,y] - [Ix,Iy] \in \mathfrak{h}_\mathbb{C}$, that is, $I[\mathrm{i}\cdot x, y] + I[x, \mathrm{i}\cdot y] + [x,y] - [\mathrm{i}\cdot x, \mathrm{i}\cdot y] \in \mathfrak{h}_\mathbb{C}$, whence $2\mathrm{i}\cdot I[x,y] + 2[x,y] \in \mathfrak{h}_\mathbb{C} = \mathrm{Ker}\,I$. This implies that $I(2\mathrm{i}\cdot I[x,y] + 2[x,y]) = 0$, whence $-2\mathrm{i}[x,y] + 2I[x,y] = 0$, and thus $[x,y] \in \mathrm{Ker}\,(I - \mathrm{i}\cdot\mathrm{id}_{\mathcal{V}_\mathbb{C}})$.
    $2°$ At this stage we just remark that, for arbitrary $\mathfrak{k} \in \mathcal{K}_0(G,H)$, the natural embedding $\mathfrak{g} \hookrightarrow \mathfrak{g}_\mathbb{C}$ induces an isomorphism of Banach spaces

$$\nu_\mathfrak{k}:\mathfrak{g}/\mathfrak{h} \to \mathfrak{g}_\mathbb{C}/\mathfrak{k}, \quad x + \mathfrak{h} \mapsto x + \mathfrak{k}.$$

In fact, since $\mathfrak{k} \cap \mathfrak{g} = \mathfrak{k} \cap \overline{\mathfrak{k}} \cap \mathfrak{g} = \mathfrak{h}_\mathbb{C} \cap \mathfrak{g} = \mathfrak{h}$, it follows that $\mathrm{Ker}\,\nu_\mathfrak{k} = \{0\}$. On the other hand, to prove that $\nu_\mathfrak{k}$ is surjective, let $z \in \mathfrak{g}_\mathbb{C}$ arbitrary. Then there exist $y_1, y_2 \in \mathfrak{k}$ such that $x = y_1 + \overline{y_2}$, whence $z = (y_2 + \overline{y_2}) + (y_1 - y_2) \in \mathfrak{g} + \mathfrak{k}$, and this shows that $z + \mathfrak{k} = (y_2 + \overline{y_2}) + \mathfrak{k} \in \mathrm{Ran}\,\nu_\mathfrak{k}$.
    $3°$ We now use the remark of stage $2°$ to define a mapping

$$\beta:\mathcal{K}_0(G,H) \to \mathcal{I}_0(G,H)$$

such that for each $\mathfrak{k} \in \mathcal{K}_0(G, H)$ the diagram

$$
\begin{array}{ccc}
\mathfrak{g}_\mathbb{C}/\mathfrak{k} & \xrightarrow{\text{mult(i)}} & \mathfrak{g}_\mathbb{C}/\mathfrak{k} \\
\nu_\mathfrak{k} \uparrow & & \uparrow \nu_\mathfrak{k} \\
\mathfrak{g}/\mathfrak{h} & \xrightarrow{\beta(\mathfrak{k})} & \mathfrak{g}/\mathfrak{h}
\end{array}
$$

is commutative, where mult(i) stands for the multiplication-by-i operator on the complex vector space $\mathfrak{g}_\mathbb{C}/\mathfrak{k}$.

It is not hard to see that the mapping β indeed takes values in $\mathcal{I}_0(G, H)$. In fact, let $\mathfrak{k} \in \mathcal{K}_0(G, H)$ arbitrary. We have $(\text{mult(i)})^2 = -\text{id}_{\mathfrak{g}_\mathbb{C}/\mathfrak{k}}$, hence $(\beta(\mathfrak{k}))^2 = -\text{id}_{\mathfrak{g}/\mathfrak{h}}$. Moreover, for each $h \in H$ we have $\text{Ad}_G(h)\mathfrak{k} \subseteq \mathfrak{k}$, hence $\text{Ad}_G(h)(\text{i} \cdot z + \mathfrak{k}) = \text{i} \cdot \text{Ad}_G(h)z + \mathfrak{k}$ for all $z \in \mathfrak{g}_\mathbb{C}$. Then $\widetilde{\text{Ad}}_G(h) \circ \beta(\mathfrak{k}) = \beta(\mathfrak{k}) \circ \widetilde{\text{Ad}}_G(h)$ for all $h \in H$, and this concludes the proof that $\beta(\mathfrak{k}) \in \mathcal{I}_0(G, H)$. (We recall from the first remark in stage 1° of the proof of Proposition 12 that $\widetilde{\text{Ad}}_G(h) = T_{p_0}(\alpha_h)$ for $h \in H$.)

4° Let $\mathfrak{k} \in \mathcal{K}_0(G, H)$ arbitrary. We are going to prove that

$$
J := \widehat{\beta(\mathfrak{k})} \in \mathcal{I}co(G, H),
$$

which will eventually show that the mapping $\widehat{\beta(\cdot)} : \mathcal{K}_0(G, H) \to \mathcal{I}co(G, H)$ is an inverse to $b_\mathcal{V}$.

We have seen in stage 3° that $\beta(\mathfrak{k}) \in \mathcal{I}_0(G, H)$, hence Proposition 10 shows that $J \in \mathcal{I}ac(G, H)$. Moreover, we have $\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \dotplus \mathcal{V}_\mathbb{C}$ and $\mathfrak{h}_\mathbb{C} = \mathfrak{k} \cap \overline{\mathfrak{k}}$, hence $\mathfrak{k} = \mathfrak{h} \dotplus (\mathcal{V}_\mathbb{C} \cap \mathfrak{k})$ and $\overline{\mathfrak{k}} = \mathfrak{h} \dotplus (\mathcal{V}_\mathbb{C} \cap \overline{\mathfrak{k}})$. Since $\mathfrak{g}_\mathbb{C} = \mathfrak{k} + \overline{\mathfrak{k}}$, it follows that

$$
\mathfrak{g}_\mathbb{C} = \mathfrak{h}_\mathbb{C} \dotplus (\mathcal{V}_\mathbb{C} \cap \mathfrak{k}) \dotplus (\mathcal{V}_\mathbb{C} \cap \overline{\mathfrak{k}}).
$$

Using this decomposition of $\mathfrak{g}_\mathbb{C}$, now define a $\mathbb{C}$-linear mapping $I : \mathfrak{g}_\mathbb{C} \to \mathfrak{g}_\mathbb{C}$ such that $\mathfrak{h}_\mathbb{C} = \text{Ker } I$, $\mathcal{V}_\mathbb{C} \cap \mathfrak{k} = \text{Ker } (I + \text{i} \cdot \text{id}_{\mathfrak{g}_\mathbb{C}})$ and $\mathcal{V}_\mathbb{C} \cap \overline{\mathfrak{k}} = \text{Ker } (I - \text{i} \cdot \text{id}_{\mathfrak{g}_\mathbb{C}})$.

In particular we have $\overline{Iz} = I\overline{z}$ for all $z \in \mathfrak{g}_\mathbb{C}$, hence $I(\mathfrak{g}) \subseteq \mathfrak{g}$. We shall check shortly that $I|_\mathfrak{g} \in \mathcal{I}_\mathcal{V}(G, H)$. In fact $(I|_\mathcal{V})^2 = -\text{id}_\mathcal{V}$ since $(I|_{\mathcal{V}_\mathbb{C}})^2 = -\text{id}_{\mathcal{V}_\mathbb{C}}$. Next note that for $h \in H$ and $z \in \mathcal{V} \cap \mathfrak{k} \subseteq \mathfrak{k}$ we have $\text{Ad}_G(h)z \in \mathfrak{k} = \mathfrak{h} \dotplus (\mathcal{V}_\mathbb{C} \cap \mathfrak{k})$ hence there exist $z_{\mathfrak{h}_\mathbb{C}} \in \mathfrak{h}_\mathbb{C}$ and $z_{\mathcal{V}_\mathbb{C}} \in \mathcal{V}_\mathbb{C}$ such that $\text{Ad}_G(h)z = z_{\mathfrak{h}_\mathbb{C}} + z_{\mathcal{V}_\mathbb{C}}$, whence

$$
I(\text{Ad}_G(h)z) = I(z_{\mathfrak{h}_\mathbb{C}} + z_{\mathcal{V}_\mathbb{C}}) = \text{i} \cdot z_{\mathcal{V}_\mathbb{C}}.
$$

Since $\text{Ad}_G(h)Iz = \text{Ad}_G(h)(\text{i} \cdot z) = \text{i} \cdot \text{Ad}_G(h)z = \text{i} \cdot z_{\mathfrak{h}_\mathbb{C}} + \text{i} \cdot z_{\mathcal{V}_\mathbb{C}}$, it follows that $I(\text{Ad}_G(h)z) - \text{Ad}_G(h)Iz \in \mathfrak{h}_\mathbb{C}$. In particular $T_1\pi \circ I \circ \text{Ad}_G(h)|_\mathcal{V} = T_1\pi \circ \text{Ad}_G(h) \circ I|_\mathcal{V}$, and thus

$$
I \in \mathcal{I}_\mathcal{V}(G, H).
$$

Also,

$$
c_\mathcal{V}(I) = \beta(\mathfrak{k}) = J_{p_0}.
$$

Thus, in order to show that $J \in \mathcal{I}co(G, H)$, we have to show that I satisfies condition (ii) in Theorem 13. To this end, it is convenient to make use of the natural projection

$$
\chi : \mathfrak{g}_\mathbb{C} = \mathcal{V}_\mathbb{C} \dotplus \mathfrak{h}_\mathbb{C} \to \mathfrak{h}_\mathbb{C}
$$

with $\operatorname{Ker}\chi = \mathcal{V}_{\mathbb{C}}$, since we have

$$Ix = \begin{cases} I(x - \chi(x)) = -\mathrm{i}(x - \chi(x)) = -\mathrm{i} \cdot x + \mathrm{i} \cdot \chi(x) \text{ if } x \in \mathfrak{k} \\ I(x - \chi(x)) = \mathrm{i}(x - \chi(x)) = \mathrm{i} \cdot x - \mathrm{i} \cdot \chi(x) \text{ if } x \in \overline{\mathfrak{k}}. \end{cases}$$

Now, using this fact, it is straightforward do show that $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] \in \mathfrak{h}_{\mathbb{C}}$ for all $x, y \in \mathfrak{g}_{\mathbb{C}}$, whence $I[Ix, y] + I[x, Iy] + [x, y] - [Ix, Iy] \in \mathfrak{h}$ for $x, y \in \mathfrak{g}$. Thus I satisfies condition (ii) in Theorem 13, and then $J \in \mathcal{I}co(G, H)$.

5° Now consider the mapping $\widehat{\beta(\cdot)}\colon \mathcal{K}_0(G, H) \to \mathcal{I}co(G, H)$, which is well defined according to stage 4°. It is easy to see that this mapping is inverse to $b_{\mathcal{V}}\colon \mathcal{I}co(G, H) \to \mathcal{K}_0(G, H)$, hence $b_{\mathcal{V}}$ is a bijection and the proof ends. $\square$

Corollary 16 ([MS98]). *Let A be a unital C^*-algebra, $n \geq 1$ an integer, and denote by $\mathrm{Fl}_{A,n}$ the set of all $n - tuples$ $(p_1, \ldots, p_n)$ of orthogonal projections in A satisfying $p_1 \leq \cdots \leq p_n$. Then $\mathrm{Fl}_{A,n}$ has a structure of complex manifold such that for every unitary element $u \in A$ the mapping*

$$\mathrm{Fl}_{A,n} \to \mathrm{Fl}_{A,n}, \quad (p_1, \ldots, p_n) \mapsto (up_1u^*, \ldots, up_nu^*),$$

is holomorphic.

Proof. Denote by U_A the unitary group of A and for each $(p_1, \ldots, p_n) \in \mathrm{Fl}_{A,n}$ define

$$\mathrm{Fl}_A(p_1, \ldots, p_n) = \{(up_1u^*, \ldots, up_nu^*) \mid u \in \mathrm{U}_A\}.$$

Clearly it suffices to equip each set $\mathrm{Fl}_A(p_1, \ldots, p_n)$ with a U_A-invariant complex structure, since $\mathrm{Fl}_{A,n}$ is the disjoint union of the sets of this form.

Next fix $(p_1, \ldots, p_n) \in \mathrm{Fl}_{A,n}$ and denote $p_0 = 0$ and

$G = \mathrm{U}_A, \quad \mathfrak{g} = \{a \in A \mid a^* = -1\},$

$H = \{u \in \mathrm{U}_A \mid up_j = p_ju \text{ for } 1 \leq j \leq n\}, \quad \mathfrak{h} = \{a \in \mathfrak{g} \mid up_j = p_ju \text{ for } 1 \leq j \leq n\}.$

Then it is clear that H is a Banach-Lie subgroup of G (since it is closed in G, the exponential map of G gives by restriction a homeomorphism of a neighborhood of $0 \in \mathfrak{h}$ onto a neighborhood of $\mathbf{1} \in H$, and $\{a \in \mathfrak{g} \mid p_jap_l = 0 \text{ for } j \neq l\}$ is a direct complement to $\mathfrak{h}$ in $\mathfrak{g}$). Moreover, denoting by

$$b \mapsto \overline{b} = -b^*$$

the conjugation of $\mathfrak{g}_{\mathbb{C}} = A$ whose fixed-point set is $\mathfrak{g}$, it follows at once that

$$\mathfrak{k} := \{b \in A \mid (p_i - p_{i-1})b(p_j - p_{j-1}) = 0 \text{ if } 1 \leq j < i \leq n\}$$

is an element of $\mathcal{K}_0(G, H)$, with the notation of Theorem 15. Then Theorem 15 implies that there exists on $G/H \simeq \mathrm{Fl}_A(p_1, \ldots, p_n)$ a G-invariant complex structure. Since $G = \mathrm{U}_A$, the proof is finished. $\square$

References

[Be03]   D. Beltiţă, *Complex homogeneous spaces of pseudo-restricted groups*, Math. Res. Lett. **10** (2003), no. 4, 459–467.

[BR04]   D. Beltiţă, T.S. Ratiu, *Symplectic leaves in real Banach Lie-Poisson spaces*, preprint math.SG/0403345.

[EF51]   B. Eckmann, A. Frölicher, *Sur l'intgrabilité des structures presque complexes*, C. R. Acad. Sci. Paris **232** (1951), 2284–2286.

[Fr55]   A. Frölicher, *Zur Differentialgeometrie der komplexen Strukturen*, Math. Ann. **129** (1955), 50–95.

[Ki76]   A.A. Kirillov, *Elements of the Theory of Representations.*, Translated from the Russian by Edwin Hewitt. Grundlehren der Mathematischen Wissenschaften, Band 220, Springer-Verlag, Berlin-New York, 1976.

[La01]   S. Lang, *Fundamentals of Differential Geometry (corrected second printing)*, Graduate texts in mathematics, vol. 191, Springer-Verlag, New York, 2001.

[Le98]   L. Lempert, *The Dolbeault complex in infinite dimensions I*, J. Amer. Math. Soc. **11** (1998), no. 3, 485–520.

[Le99]   L. Lempert, *The Dolbeault complex in infinite dimensions II*, J. Amer. Math. Soc. **12** (1999), no. 3, 775–793.

[Le00]   L. Lempert, *The Dolbeault complex in infinite dimensions III. Sheaf cohomology in Banach spaces*, Invent. Math. **142** (2000), no. 3, 579–603.

[MS97]   M. Martin, N. Salinas, *Flag manifolds and the Cowen-Douglas theory*, J. Operator Theory **38** (1997), no. 2, 329–365.

[MS98]   M. Martin, N. Salinas, *The canonical complex structure of flag manifolds in a C^*-algebra*, in: Nonselfadjoint operator algebras, operator theory, and related topics, Oper. Theory Adv. Appl., vol. 104, Birkhäuser, Basel, 1998, pp. 173-187.

[NRW01] L. Natarajan, E. Rodríguez-Carrington, J.A. Wolf, *The Bott-Borel-Weil theorem for direct limit groups*, Trans. Amer. Math. Soc. **353** (2001), no. 11, 4583–4622.

[Ne04]   K.-H. Neeb, *Infinite-dimensional groups and their representations*, in: Lie Theory, Progr. Math., vol. 228, Birkhäuser, Boston, MA, 2004, pp. 213–328.

[NN57]   A. Newlander, L. Nirenberg, *Complex analytic coordinates in almost complex manifolds*, Ann. of Math. (2) **65** (1957), 391–404.

[Nu92]   F. Nübel, *On integral manifolds for vector space distributions*, Math. Ann. **294** (1992), no. 1, 1–17.

[Pa00]   I. Patyi, *On the $\overline{\partial}$-equation in a Banach space*, Bull. Soc. Math. France **128** (2000), no. 3, 391–406.

[Up85]   H. Upmeier, *Symmetric Banach Manifolds and Jordan C^*-algebras*, North-Holland Mathematics Studies, 104. Notas de Matemática, 96. North-Holland Publishing Co., Amsterdam, 1985.

[Wi90]   D.R. Wilkins, *The Grassmann manifold of a C^*-algebra*, Proc. Roy. Irish Acad. Sect. A **90** (1990), no. 1, 99–116.

[Wi94]   D.R. Wilkins, *Infinite dimensional homogeneous manifolds*, Proc. Royal Irish Acad. Sect. A **94** (1994), no. 1, 105–118.

Institute of Mathematics "Simion Stoilow" of the Romanian Academy, P.O. Box 1-764, RO-014700 Bucharest, Romania

E-mail address: Daniel.Beltita@ imar.ro